On Bounds of Shift Variance in Two-Channel Multirate Filter Banks

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On Bounds of Shift Variance in Two-Channel Multirate Filter Banks

Til Aach, Senior Member, IEEE, and Hartmut Führ

Abstract—Critically sampled multirate FIR filter banks exhibit periodically shift variant behavior caused by non-ideal anti-aliasing filtering in the decimation stage. We assess their shift variance quantitatively by analysing changes in the output signal when the filter bank operator and shift operator are interchanged. We express these changes by a so-called commutator. We then derive a sharp upper bound for shift variance via the operator norm of the commutator, which is independent of the input signal. Its core is an eigensystem analysis carried out within a frequency domain formulation of the commutator, leading to a matrix norm which depends on frequency. This bound can be regarded as a worst case instance holding for all input signals. For two channel FIR filter banks with perfect reconstruction (PR), we show that the bound is predominantly determined by the structure of the filter bank rather than by the type of filters used. Moreover, the framework allows to identify the signals for which the upper bound is almost reached as so-called near maximizers of the frequency-dependent matrix norm. For unitary PR filter banks, these near maximizers are shown to be narrow-band signals. To complement this worst-case bound, we derive an additional bound on shift variance for input signals with given amplitude spectra, where we use wide-band model spectra instead of narrow-band signals. Like the operator norm, this additional bound is based on the above frequency-dependent matrix norm. We provide results for various critically sampled two-channel filter banks, such as quadrature mirror filters, PR conjugated quadrature filters, wavelets, and biorthogonal filters banks.

Index Terms—Multirate filters, critical sampling, operator norm, modulation vectors, eigensystem analysis, uniform bounds, perfect reconstruction, unitary filter banks, biorthogonal filter banks.

I. INTRODUCTION

Amplifying rate conversion in multirate filter systems employing linear shift-invariant (LSI) filters generally implies that these systems are linear periodically shift variant (LPSV) rather than LSI [1]–[3]. Examples are fractional sampling rate converters [1], [4], [5], the Gaussian and Laplacian pyramid [6], block and lapped transforms [7]–[9] or multirate filter banks [10]–[16]. The processing results of, for instance, subband filtering [17] or subband quantization [18]–[20], [21], p.126], thus depend on shifts of the input signal [3], [19], [22], [23], and often influence other processing stages, such as motion estimation [24]. Multirate filter banks are also used as efficient convolvers [25], [26], especially for broadband chirp signals of long duration in digital radar receivers [27]. Thresholding of low-magnitude subband coefficients [27] as well as subband rounding effects [28], p.129] make such systems LPSV. In the time domain, LPSV systems are characterized by a periodically varying impulse response while in the frequency domain, they can be described by two-dimensional spectra known as bifrequency maps [1], [2], [29]. As shown by Vaidyanathan and Mitra in [3], LPSV systems can generally be represented by critically sampled polyphase networks. Approaches to reduce or prevent periodic shift variance include the use of complex wavelets [21], [23], [30], cycle spinning [31], finding the best wavelet representation after decomposing an input signal for all circular shifts [32], using wavelet frames [16], or employing two parallel biorthogonal wavelet transforms where the wavelets are related by the Hilbert transform [33] (cf. also [23], [34]). These approaches generally imply overcomplete representations. This contribution puts forward a quantitative assessment of shift variance in critically sampled filter banks, where we start with a general problem formulation, and then focus on orthogonal and biorthogonal two-channel filter banks. Since this analysis assumes deterministic signals, let us first briefly point out relations to the statistical point of view, specifically the behavior of LPSV systems with wide sense stationary (WSS) input signals.

When the input signal is a WSS random signal, LPSV systems may introduce cyclic nonstationarities into the processing result [29], [35]–[39], making the output signal wide sense cyclostationary (WSCS). Measures such as the nonaliasing energy ratio (NER) [40] or mean square error average over the cyclic fluctuations of the correlation structure, thus implicitly assuming that the output signal is WSS. As shown by Signoroni and Leonardi for 3D wavelet compression of biomedical 3D data sets or video data [41], [42], the periodic variations of the peak-signal-to-noise ratio (PSNR) in the reconstructed signal may well cause relevant visible effects. Similarly, Seidner shows the effects of cyclostationarities introduced into originally stationary noise when resampling images [5, p.1886]. While shift variance in a multirate system is generated by aliasing occurring in the decimation stage, cyclic nonstationarities are caused by non-ideal anti-imaging filtering in the interpolation stage [4], [13], [19], [35], [37], [38], [41]. Periodic shift variance and the generation of cyclic nonstationarities in multirate LPSV systems are closely related [29], [37], especially for paraunitary and biorthogonal perfect reconstruction (PR) filter banks [43]. In earlier work, we therefore analyzed the behavior of multirate filter bank channels with respect to periodic shift variance and generation of WS-
cyclostationarity in a parallel, comparative manner, and gave a quantitative comparison for various filter sets used in PR filter banks as well as for block transforms [43]. While such a comparative analysis highlights the dual behavior of shift variance and cyclostationarity, it focussed on energy spectra and power spectra of deterministic and stationary random signals, respectively. Also, only individual filter bank channels were considered, with the extension to multirate filter banks requiring that cross spectra be also taken into account.

In his work on complex wavelets, Kingsbury quantifies shift dependence by the ratio of the energy of the aliasing components in the filter bank transfer equation to the energy of the non-aliased components [23, Eqs. (1),(5)]. While allowing to rank different filter banks with respect to their shift variant behavior, and in particular showing that for decimation by a factor of two, the complex filters are perfectly shift invariant, this measure does not provide a direct relation to changes in the (processed) output signal as a consequence of shifting the input. Kingsbury additionally shows that, besides reduc-

tion of aliasing, the redundancy of the dual-tree complex wavelet transform enables the design of directionally selective filters for multidimensional signals, similarly as for block transforms in [8], [44], [45]. With separable, critically sampled real-valued filters, this cannot be achieved.

In this paper, we provide a comprehensive quantitative framework for the analysis of shift variance in multirate filter banks, which evaluates changes in the output signal when the filter channel operator and shift operator are interchanged. The changes are described by a so-called commutator, which expresses the difference signal between the filter bank response to a shifted input signal and the shifted filter bank output. We derive a sharp upper bound for shift variance via the operator norm of the commutator, which is independent of the input signal. Its core is an eigensystem analysis carried out in a frequency-domain formulation of the energy of the difference signal, or equivalently, the \( L_2 \)-norm of the commutator, where the eigensystem analysis provides a frequency dependent matrix norm. This bound holds for all input signals simultaneously, and can be understood as describing the worst-case behavior of each channel. We then focus on two-channel FIR filter banks and show that, for PR FIR filter banks, this bound is identical in both channels, and always greater than or equal to one. For unitary PR filter banks, the bound is even always equal to one, thus being determined by the property of orthogonality rather than by the specific filter set used. Moreover, we discuss the corresponding worst-case signals or “near maximizers”. For a unitary PR filter bank, these near maximizers are narrow-band signals around the transition frequency between its two channels. Therefore, to complement this worst-case bound, and to capture more of the filter properties rather than the influence of the filter bank structure, we develop an additional bound for shift variance under the assumption that the filter bank input signals exhibit given amplitude spectra. We use here wide-band model spectra instead of narrow-band signals. Like the uniform bound, this additional bound is based on the above frequency-dependent matrix norm. We provide results for variety of critically

\[
s[n] \quad H(z) \quad M \quad M \quad G(z) \quad y[n]
\]

Fig. 1. One channel of a multirate filter bank.

sampled complete multirate filter banks, such as quadrature mirror filters, PR conjugated quadrature filters, wavelets, and biorthogonal filters banks.

By design, PR filter banks are shift invariant if the sub-bands are not processed. Subband processing, however, alters generally also the aliasing in the subbands. The aliasing contributions of the filter bank channels then do not cancel anymore, thus making the output shift variant [23, p.250], [21, p.126]. We therefore extend our analysis towards two-channel PR filter banks with an elementary form of subband processing. We conclude with results for a variety of unitary filters, such as quadrature mirror (QM) filters with linear and nonlinear phase, multiplierless QM filters, and biorthogonal filter pairs of different lengths.

Fig. 1 shows a channel of a multirate filter bank, consisting of a downsampler followed by an upsampler placed between analysis and synthesis filters \( H(z) \) and \( G(z) \). The input signal is denoted by \( s[n] \), \( s : Z \rightarrow C \), and is assumed to be of finite energy, i.e., \( s \in L^2(Z) \), where \( L^2(Z) \) is the space of square-summable doubly-infinite sequences of complex numbers. The \( z \)-transform of \( s[n] \) is \( S(z) \), and its norm (or energy) is

\[
E_s = ||s||_2^2 = \sum_{n=-\infty}^{\infty} |s[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(e^{j\omega})|^2 d\omega .
\]

The output signal of the filter bank channel with \( s[n] \) at its input is \( y[n] \). When shifting \( s[n] \) by \( m \) samples to \( s[n-m] \), the output is denoted by \( x^m[n] \), while \( y^m[n] = y[n-m] \) is the output, shifted by \( m \) samples, in response to \( s[n] \). The modulation vector of length \( M \) of \( s[n] \) is defined by \( s_M(z) = [S(z), S(ze^{j\pi/M}), \ldots, S(ze^{j(M-1)\pi/M})]^T \), where the superscript \( T \) denotes transposition, and \( W = e^{-j2\pi/M} \). The adjoint of a vector \( x \) is \( 1 \in C^M \) or a matrix \( T \in C^{M \times M} \) is denoted by \( x^* = \overline{x}^T \) and \( T^* = \overline{T}^T \), respectively, where the overline stands for complex conjugation. The energy \( E_s \) can then also be expressed as

\[
E_s = \frac{1}{2\pi} \int_{-\pi}^{\pi/M} s_M^*(e^{j\omega}) s_M(e^{j\omega}) d\omega .
\]

II. NORM-BASED ANALYSIS OF SHIFT VARIANCE

A. Channel operators, shift operators, and commutators

To quantify the behavior of a filter bank channel with respect to shifts, we define the shift operator \( \tau_m : L^2(Z) \rightarrow L^2(Z) \) by \( \tau_m(s[n]) = s[n-m] \); hence, \( y^m[n] = \tau_m(y[n]) \). Describing the filter bank channel by an operator \( K : L^2(Z) \rightarrow L^2(Z) \), we have \( y[n] = K(s[n]) \). The channel output \( x^m[n] \) in reaction to the shifted input signal \( s[n-m] \) is in operator notation given by \( x^m = (K \circ \tau_m)(s) \), where \( \circ \) stands for the concatenation
of operators. Vice versa, the output signal \( y^m[n] \) shifted after passing \( s[n] \) through the filter bank is \( y^m = (\tau_m \circ K)(s) \). We call an operator \( T : \ell^2(Z) \rightarrow \ell^2(Z) \) fully shift invariant if, for all \( m \in \mathbb{Z} \) and for all \( s \in \ell^2(Z) \), \( (T \circ \tau_m)(s) = (\tau_m \circ T)(s) \). This can equivalently be expressed in terms of the commutator \([46]\) defined as \( [T, \tau_m] = T \circ \tau_m - \tau_m \circ T \); then, the channel operator \( K \) is fully shift invariant if and only if \([K, \tau_m](s) = 0 \) for all \( m \) and \( s[n] \). Equivalently, when denoting by \( r^m[n] = x^m[n] - y^m[n] \) the difference (or "residual") signal between \( x^m[n] \) and \( y^m[n] \), we have \( r^m = [K, \tau_m](s) \) and \( K \) is fully shift invariant when \( r^m[n] = 0 \) for all \( m \), \( n \) and \( s \). Of course, due to the non-ideal band-limiting properties of the filters \( h[n] \) and \( g[n] \), \( K \) is not fully shift invariant. Rather, \( K \) is periodically shift variant with period \( M \) \([2], [3]\). To measure its degree of shift variance for a given \( m \), we start out from the energy \( E_r[m] = \|r^m\|^2 \) of \( r^m \), which is identical to the squared \( L_2 \)-norm \( \| [K, \tau_m](s) \|^2 \) of the commutator output to the input signal \( s \); or, in other words, proportional to the mean square error (or Euclidean distance) between \( x^m[n] \) and \( y^m[n] \).

**B. The filter bank channel**

The \( z \)-transform \( Y(z) \) of \( y[n] \) is given by \([15]\)

\[
Y(z) = \frac{G(z)}{M} \sum_{k=0}^{M-1} H(zW^k)S(zW^k) = \frac{G(z)}{M} \hat{h}_M^T(z)M(z) \tag{3}
\]

where \( h_M(z) \) is the modulation vector of \( h[n] \). The modulation vector \( y_M(z) \) is

\[
y_M(z) = \frac{1}{M} g_M(z) \hat{h}_M^T(z)M(z) \tag{4}
\]

with \( g_M(z) \) being the modulation vector of \( g[n] \). In operator notation, we have

\[
y_M(z) = \hat{K}_M(s_M(z)) \tag{5}
\]

with \( \hat{K}_M : \mathbb{C}^M \rightarrow \mathbb{C}^M \) being the channel operator mapping the modulation vector \( s_M(z) \) to \( y_M(z) \) according to (4).

For the shifted input signal \( \tau_m[s[n]] = s[n-m] \), the \( z \)-transform \( X^m(z) \) of the output \( x^m[n] \) is

\[
X^m(z) = \frac{z^{-m}}{M} \sum_{k=0}^{M-1} H(zW^k)W^{-mk}S(zW^k) \tag{6}
\]

where \( D \in \mathbb{C}^{M \times M} \) is the diagonal matrix \( \text{diag} [1, W, W^2, \ldots, W^{M-1}] \), and consequently, \( D^{-m} = \text{diag} [1, W^{-m}, \ldots, W^{-m(M-1)}] \). The modulation vector \( x_M^m(z) \) of \( x^m[n] \) then is

\[
x_M^m(z) = \frac{z^{-m}}{M} g_M(z) \hat{h}_M^T(z)D^{-m}M(z) \tag{7}
\]

Shifting the output \( y[n] \) in response to the input \( s[n] \) yields the \( z \)-transform \( Y^m(z) = z^{-m}Y(z) \), and the modulation vector \( y_M^m(z) \) of the shifted output \( y^m[n] \) thus is

\[
y_M^m(z) = \frac{z^{-m}}{M} D^{-m}g_M(z)\hat{h}_M^T(z)M(z) \tag{8}
\]

The modulation vector \( y_M^m(z) \) of the difference signal \( r^m[n] \) then is

\[

\begin{align*}
\mathbf{r}_M^m(z) &= x_M^m(z) - y_M^m(z) \tag{9} \\
&= \frac{z^{-m}}{M} g_M(z) \hat{h}_M^T(z)D^{-m} - D^{-m}g_M(z)\hat{h}_M^T(z)M(z) \tag{10} \\
&= T(m, z)s_M(z) ,
\end{align*}

\]

with the matrix \( T(m, z) \in \mathbb{C}^{M \times M} \) being given by

\[
T(m, z) = \frac{z^{-m}}{M} [g_M(z)\hat{h}_M^T(z)D^{-m} - D^{-m}g_M(z)\hat{h}_M^T(z)] . \tag{11}
\]

**C. Residual Signal and Commutator Norm**

We can now calculate the energy of the difference signal \( r^m[n] \) for a given \( s[n] \). With the modulation vector \( y^m_M[z] \), \( E_r[m] \) is

\[
E_r[m] = \|r^m[z]\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |R^m(e^{j\omega})|^2d\omega \tag{12}
\]

With (10), we obtain for the norm of \( \mathbf{r}_M^m(z) \)

\[
\|\mathbf{r}_M^m(z)\|^2 = \|(r_M^m(z))^T \mathbf{r}_M^m(z)\| = s_M^T(z)T^*(m, z)T(m, z)s_M(z) \tag{13}
\]

where the matrix \( A \in \mathbb{C}^{M \times M} \) is given by \( A(m, z) = T^*(m, z)T(m, z) \). \tag{14}

Thus, the energy \( E_r[m] \) in (12) can also be expressed as

\[
E_r[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_M^T(e^{j\omega})A(m, e^{j\omega})s_M(e^{j\omega})d\omega \tag{15}
\]

where \([K, \tau_m](s)\) is the commutator as defined in section II-A. As already stated above, \( \| [K, \tau_m](s)\|^2 \) is proportional to the mean square error between \( x^m[n] \) and \( y^m[n] \).

**D. Operator Norm: Uniform Bounds on Shift Variance**

So far, our analysis depends on the input signal \( s \). For a given \( s \), (15) is not necessarily more manageable or more easily evaluated than the original definition in (12). However, (15) does allow to derive sharp uniform bounds, which hold simultaneously for all input signals. More precisely, the quantity we are after is given by

\[
E[m] = \sup_r E_r[m] , \tag{16}
\]

where \( r \) ranges over all residual signals arising from any input signal \( s \) with \( \|s[n]\|_2 \leq 1 \). This quantity is called the (squared)
operator norm of the commutator \([K, r_m]_\infty\), and is denoted by
\[
\| [K, r_m]_\infty \| \supseteq \| E[m] \| = \sup \{ \| [K, r_m]_\infty \| \| s \| \leq 1 \} = \| [K, r_m]_\infty \| . 
\]
(17)
As an immediate consequence of linearity, one then obtains for all input signals \( s \) with associated residual \( r \) that
\[
E_r[m] = \| [K, r_m]_\infty \| \| s \| \leq \| [K, r_m]_\infty \| \| s \| = E[m] \| s \| ,
\]
(18)
Hence, the energy of the residual can be estimated by a constant multiple of the input energy, and by definition, \( E_r[m] \) is the smallest possible constant satisfying this inequality.

Let us briefly mention that one upper bound of \( E_r[m] \) is easily derived, in the form of
\[
E_r[m] = \| K \circ r_m - r_m \circ K \|_\infty \leq 2 \| K \| \| s \| ^2, 
\]
(19)
where we used the triangle inequality and \( \| S \circ T \| \leq \| S \| \| T \| \) for arbitrary linear operators \( S, T \), as well as \( \| r_m \| = 1 \) (see [46] for the relevant properties of the operator norm). \( \| K \| \) denotes the operator norm of the channel operator \( K \). Of course, this bound is not sufficiently distinctive for our purposes.

We now proceed to determine the supremum \( E \) in two steps. First, we seek the supremum of the integrand of (15) for every \( \omega \in [-\pi/M, \pi/M] \). For \( T \in C^{M \times M} \), this is given by the matrix norm
\[
\| T \|_\infty = \sup_{|s_{\omega}| = 1} |s_{\omega}^T T s_{\omega}| = \sqrt{\sup_{|s_{\omega}| = 1} |s_{\omega}^T A s_{\omega}|}.
\]
(20)
Note that we have used the same notation for matrix and operator norm; if we identify matrices \( T \in C^{M \times M} \) in the usual way with linear operators \( C^M \to C^M \), then
\[
\| T \|_\infty = \sup_{|s_{\omega}| = 1} |s_{\omega}^T T s_{\omega}|, 
\]
in complete analogy to (17). For matrices, the supremum in (20) is, in fact, a maximum. It is given by
\[
\| T \|_\infty = \sup_{|s_{\omega}| = 1} |s_{\omega}^T T s_{\omega}| = \sqrt{\sup_{|s_{\omega}| = 1} |s_{\omega}^T A s_{\omega}|} = \sqrt{\lambda_\ast (A(m, e^{j\omega}))},
\]
(21)
where \( \lambda_\ast (A(m, e^{j\omega})) \) is the largest eigenvalue of \( A(m, e^{j\omega}) \).

Note that \( A = T^* T \) has no nonnegative real eigenvalues. Since \( T \) is a matrix of rank at most two, \( A \) has at most two nonzero eigenvalues. With (14), the eigenvalue \( \lambda_1 \) is equal to the square of the singular value \( \sigma_1 \) of \( T^* \), and thus,
\[
\sqrt{\lambda_1 (m, e^{j\omega})} = \sigma_1 (m, e^{j\omega}).
\]
Furthermore, the eigensystem analysis of \( A(m, e^{j\omega}) \) also provides the modulation vector \( e_{M1}(e^{j\omega}) \) maximizing the integrand of (15), which is given by the normalized eigenvector of \( A(m, e^{j\omega}) \) corresponding to \( \lambda_1 (m, e^{j\omega}) \).

With the supremum of the integrand in (15) being given by the matrix norm \( \| T \|_\infty \) in (21), we can, in a second step, determine the operator norm \( \| [K, r_m]_\infty \|_\infty \) of the commutator in (17), i.e., the supremum of \( E_r[m] \) as defined in (16). This boils down to finding extremal values of the right-hand side of (21). For the formulation of results that are both sharp and general, the following somewhat technical definition is useful:

**Definition 1:** Let \( f : (-\pi/M, \pi/M) \to \mathbb{R} \) be a Borel-measurable function. The essential supremum of \( f \) is defined as
\[
\text{ess sup}_{\omega \in [-\pi/M, \pi/M]} f(\omega) = \inf \{ t \in \mathbb{R} : f(\omega) \leq t \text{ for almost every } \omega \in [-\pi/M, \pi/M] \}.
\]
See [47] for properties of the essential supremum. Informally, the essential supremum can be regarded as the supremum of \( f(\omega) \), obtained after possibly ignoring \( \omega \) from a set of measure zero. Replacing the supremum by the essential supremum often allows sharper estimates for the integral. We note that whenever \( f \) is continuous on \([-\pi/M, \pi/M]\), as it will be the case here for FIR filters, the essential supremum is in fact the maximum of \( f \).

**Theorem 2:** The operator norm \( E[m] \) is given by
\[
E[m] = \| [K, r_m]_\infty \|_\infty = \text{ess sup}_{\omega \in [-\pi/M, \pi/M]} \lambda_1 (m, e^{j\omega}).
\]
(22)
Whenever \( G \) and \( H \) are continuous, e.g., if \( g, h \) are FIR filters, then
\[
E[m] = \| [K, r_m]_\infty \|_\infty = \max_{\omega \in [-\pi/M, \pi/M]} \lambda_1 (m, e^{j\omega}).
\]
(23)
**Proof.** To see that (22) is correct, let \( C \) denote the right-hand side of (22). Then, by definition of the essential supremum, and with (21),
\[
\| T(m, e^{j\omega}) \|_\infty \leq C 
\]
is true for almost all \( \omega \) (or, when the essential supremum is a maximum, for all \( \omega \)). Integration over \( \omega \) then yields with (15)
\[
E_r[m] \leq \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \lambda_1 (m, e^{j\omega}) \| s_M(e^{j\omega}) \|_\infty \| s_M(e^{j\omega}) \|_\infty d\omega
\]
(24)
\[
\leq \frac{C}{2\pi} \int_{-\pi/M}^{\pi/M} \| s_M(e^{j\omega}) \|_\infty \| s_M(e^{j\omega}) \|_\infty d\omega 
\]
(25)
and thus with Parseval’s theorem
\[
E_r[m] \leq C \| s \|_2^2.
\]
(26)
On the other hand, pick any \( \epsilon > 0 \). We now show the existence of a signal \( s \) with
\[
E_r[m] > (C - \epsilon) \| s \|_2^2.
\]
By definition of the essential supremum, the set \( A = \{ \omega \in [-\pi/M, \pi/M] : \lambda_1 (m, e^{j\omega}) > C - \epsilon \} \) is a measurable set of positive measure (if \( \lambda_1 (m, \cdot) \) is continuous, e.g., for FIR filters, \( A \) is in fact open and nonempty, hence a disjoint union of nonempty open intervals, all of which have nonzero length). For \( \omega \in A \), define \( s_M(e^{j\omega}) = e_{M1}(e^{j\omega}) \); for \( \omega \notin A \), let \( s_M(e^{j\omega}) = 0 \). Since the matrices \( T(m, e^{j\omega}) \) depend measurably on \( \omega \), this defines a measurable family of vectors satisfying
\[
\frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \| s_M(e^{j\omega}) \|_\infty \| s_M(e^{j\omega}) \|_\infty d\omega = \frac{|A|}{2\pi}.
\]
(27)
This equality is due to the fact that \( \|s_M(e^{j\omega})\|_2 = 1 \forall \omega \in A \). \(|A|\) is called the Lebesgue measure of \( A \) [47]; in the case where \( A \) is a disjoint union of open intervals, \(|A|\) is the sum of the interval lengths. For \( s_M(e^{j\omega}) \) as above, (15) becomes

\[
\frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} s_M^*(e^{j\omega}) A(m, e^{j\omega}) s_M(e^{j\omega}) d\omega
\]

\[
> \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} (C - \epsilon) s_M^*(e^{j\omega}) s_M(e^{j\omega}) d\omega
\]

\[
= (C - \epsilon) \frac{|A|}{2\pi}.
\]

From \( s_M(e^{j\omega}) \) specified as above, we obtain a signal \( s \) by inverse Fourier transform. By (27) and with (28), we have \( \|s\|_2^2 = \frac{|A|}{2\pi} \), and then (28) implies (26).

Taken together, the inequalities (25) and (26) prove equality (22).

Note that in general we cannot expect to find a “worst-case” signal \( s \) such that \( E_s[m] = E[m] \), with \( r \) being the residual associated to \( s \). Such signals only exist if

\[
\lambda_1(m, e^{j\omega}) = E[m]
\]

holds for all \( \omega \) in a set of positive measure or, informally, on an interval of non-vanishing length (recall that we consider only finite energy signals). We will usually have to settle for near maximizers \( s \), i.e., signals \( s \) satisfying \( E_s[m] \geq E[m] - \epsilon \) for some fixed \( \epsilon \). The proof of (22) shows how to find those signals on the Fourier transform side, by specifying \( s_M(e^{j\omega}) \) to be the eigenvector of \( A(m, e^{j\omega}) \) associated to \( \lambda_1(m, e^{j\omega}) \), but only for those frequencies for which this eigenvalue is close to \( E[m] \) as specified in (22), (23). Thus the analysis of near maximizers for \( E_s[m] \) allows to identify critical frequency bands. On signals with low energy in those frequency bands, the channel operators are almost shift invariant.

III. TWO-CHANNEL FILTER BANKS

We now turn towards the two-channel filter bank with analysis filters \( h_0[n] \), \( h_1[n] \) and synthesis filters \( g_0[n] \), \( g_1[n] \) in Fig. 2. Since the filter bank is periodically variant with period \( M = 2 \), it suffices to evaluate (21) for \( m = 1 \). For simplicity, we will omit the variable \( m \) throughout this section.

In the following, we first apply our concept to each channel individually, where we consider channels of PR filterbanks in general. After this, we discuss unitary PR filter banks. We then examine operators obtained by elementary subband processing in a PR filter bank.

A. Single Channel Analysis

Throughout this subsection, \( i \in \{0, 1\} \). Let \( K_i \) denote the channel operator associated to \( g_i, h_i \). With the modulation vectors

\[
h_{iM}(z) = [H_i(z), H_i(-z)]^T, \quad g_{iM}(z) = [G_i(z), G_i(-z)]^T
\]

and

\[
D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

(11) yields after simplification

\[
T_i(z) = z^{-1} \begin{bmatrix} 0 & -G_i(z)H_i(-z) \\ G_i(-z)H_i(z) & 0 \end{bmatrix},
\]

resulting in

\[
\lambda_1(e^{j\omega}) = \max(|G_i(e^{j\omega})H_i(e^{j\omega})|^2, |G_i(e^{j\omega})H_i(-e^{j\omega})|^2).
\]

Finally, we obtain for the operator norm:

\[
\|[K_i, \tau_i]\|_\infty = \sup_{\omega \in [-\pi, \pi]} |G_i(e^{j\omega})H_i(e^{j\omega})|.
\]

Evidently, (31) has an intuitive interpretation. The usual purpose of a filter bank is the separation of frequency bands; in the case of two channels, these are commonly a low-frequency band and a high-frequency band. Accordingly, the pair \( G_i, H_i \) will be concentrated in a frequency band of length \( \pi \). In particular, the frequencies \( \omega, \omega + \pi \) will usually belong to different frequency bands, and thus at least one of the factors in the product \( G_i(e^{j\omega})H_i(-e^{j\omega}) \) should be small (cf. the discussion in [23, sec. 4]). If \( H_i, G_i \) are lowpass or highpass, and \( \omega \) is close to 0 or \( \pi \), then the product will be small. However, if \( H_i, G_i \) are continuous functions, there has to be a transitional region where \( H_i \) and \( G_i \) take intermediate values. This is where the maximum value of \( \lambda_1(e^{j\omega}) \) is to be expected. Accordingly, recalling the discussion of near maximizers in subsection II-D, the Fourier transforms of near maximizers \( s \) of \( \|[K_i, \tau_i](s)\|_2 \) will be concentrated in the transitional region, i.e., at the boundaries of the two frequency bands that the channels are supposed to separate. In critically sampled unitary two-channel PR filter banks, this region is located around \( \pm \pi/2 \), as proved in section III-C and illustrated in section V. Let us emphasize that these as well as the following considerations hold for filters with continuous spectra, such as FIR filters. Designing \( H_i(z) \) and \( G_i(z) \) as ideal lowpass and highpass filters, respectively, with discontinuous transitions from passband to stopband at the cutoff frequency \( \omega_c = \pi/2 \), eliminates aliasing, and implies IIR filters with sinc-shaped impulse responses. The filter bank channel is in this ideal case fully shift invariant.

B. Single Channels of PR Filter Banks

Now assume that the channels yield perfect reconstruction, meaning \( s = K_0s + K_1s \) for all \( s \). (Without loss of generality, we may omit a shift between input and output of the filter
bank, as the eigenvalues in (31) depend only on amplitude spectra.) We first recall that the reconstruction filters $g_0, g_1$ are then uniquely determined by $h_0, h_1$, and derive an expression for the commutator norms $\| [K_1, \tau_1] \|_{\infty}$ in terms of $h_0, h_1$. We prove that this bound is always greater than or equal to one, and that for unitary filter banks, it is equal to one.

By (8) (with $m = 0$), the perfect reconstruction property translates to the equation

$$s_M(z) = \frac{1}{2} (g_0M(z)h_0M(z)^T + g_1M(z)h_1M(z)^T) s_M(z),$$

for almost all $z$, and all signals $s$. In matrix notation, we obtain

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = G(z)H(z)$$

(33)

with the modulation matrices [14] (or alias component matrices, [15]) $G(z)$ and $H(z)$ given by

$$G(z) = \begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}$$

(34)

and

$$H(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}.$$  

(35)

Thus, $G(z)$ is uniquely determined by $H(z)$, and $2 \times 2$ matrix inversion yields [14, p. 113] [48, sec. 6.3], [49]

$$G_0(z) = \frac{2H_1(-z)}{\det(H(z))},$$

$$G_1(z) = \frac{-2H_0(-z)}{\det(H(z))}.$$ 

(36)

Plugging these expressions into (31) results in

$$\lambda_{01}(e^{j\omega}) = \max \left( \frac{2|H_1(e^{j\omega})H_0(e^{j\omega})|^2}{|\det(H(-e^{j\omega}))|^2}, \frac{2|H_1(-e^{j\omega})H_0(-e^{j\omega})|^2}{|\det(H(e^{j\omega}))|^2} \right) = \lambda_{11}(e^{j\omega}),$$

(37)

and (32) provides

$$\| [K_1, \tau_1] \|_{\infty} = \sup_{\omega \in [-\pi, \pi]} \frac{2|H_0(e^{j\omega})H_1(e^{j\omega})|}{|\det(H(-e^{j\omega}))|^2}.$$  

(38)

In particular, we make the slightly surprising observation that

$$\| [K_0, \tau_1] \|_{\infty} = \| [K_1, \tau_1] \|_{\infty}.$$  

(39)

A further useful observation concerns the matrix $T_i(e^{j\omega})$. We just noted that the eigenvalues $\lambda_{01}, \lambda_{11}$ of $A_i(e^{j\omega}) = T_i^*(e^{j\omega})T_i(e^{j\omega})$ are given by (38)

$$\frac{4|H_0(e^{j\omega})H_1(e^{j\omega})|^2}{|\det(H(-e^{j\omega}))|^2}, \frac{4|H_0(-e^{j\omega})H_1(-e^{j\omega})|^2}{|\det(H(e^{j\omega}))|^2}.$$  

By definition of the matrix $H(e^{j\omega})$, we have

$$\det(H(e^{j\omega})) = -\det(H(-e^{j\omega}))$$

(40)

Hence the two eigenvalues coincide if and only if

$$|H_0(e^{j\omega})H_1(e^{j\omega})| = |H_0(-e^{j\omega})H_1(-e^{j\omega})|$$

(41)

holds. In this setting, the matrix $T_i(e^{j\omega})$ is a multiple of an isometry.

This observation has the following relevance for near maximizers: Recall from the discussion at the end of subsection II-D that near maximizers are found by specifying $s_M(e^{j\omega})$ as eigenvector associated to $\lambda_1(m, e^{j\omega})$, whenever $\lambda_1(m, e^{j\omega})$ is close to $E[m]$, and zero elsewhere. If condition (41) is fulfilled for all $\omega$, then any (nonzero) choice of $s_M(e^{j\omega})$ provides such an eigenvector.

**Theorem 3:** If $H_i, G_i$ are continuous functions, for $i = 0, 1$, then we can estimate the commutator norms as follows:

$$\| [K_1, \tau_1] \|_{\infty} \geq 1.$$  

(42)

**Proof.** We first choose $\omega \in [-\pi, \pi]$ such that $|H_0(e^{j\omega})| = |H_0(-e^{j\omega})|$. If $|H_0(1)| = |H_0(-1)|$, we may choose $\omega = 0$; otherwise, the continuous function $\omega \mapsto |H_0(e^{j\omega})| - |H_0(-e^{j\omega})|$ changes its sign between 0 and $\pi$, and the intermediate value theorem yields a suitable $\alpha \in (0, \pi)$ with $\alpha = |H_0(e^{j\omega})| = |H_0(-e^{j\omega})|$. Note that since $H(e^{j\omega})$ is invertible, $\alpha \neq 0$. We then obtain

$$\left| \frac{\det(H(e^{j\omega}))}{\det(H(-e^{j\omega}))} \right| = \left| \frac{\alpha \theta_1 H_1(e^{j\omega}) - \alpha \theta_2 H_1(-e^{j\omega})}{\theta_1 H_1(-e^{j\omega}) - \theta_2 H_1(e^{j\omega})} \right|^2$$

with suitable complex numbers $\theta_1, \theta_2$ of modulus one. Then the triangle inequality implies

$$\max(|H_1(e^{j\omega})|, |H_1(-e^{j\omega})|) \geq \frac{|\det(H(e^{j\omega}))|}{2\alpha}.$$ 

We may assume that the greater value of the two is attained at $e^{j\omega}$ (otherwise replace $\omega$ by $\omega + \pi$). We then obtain the estimate

$$\frac{2|H_0(e^{j\omega})H_1(e^{j\omega})|}{|\det(H(-e^{j\omega}))|^2} \geq \frac{2\alpha |\det(H(e^{j\omega}))|}{|\det(H(-e^{j\omega}))|^2} = 1,$$

using (40). Since $H_0$ and $H_1$ are assumed continuous, it follows that the essential supremum is a maximum, hence must be $\geq 1$ as well. This proves (42).

**C. Single Channels of Unitary Filter Banks**

We retain the notations of the previous subsection, and now turn to the case where the matrices $H(e^{j\omega})$ are unitary. Then the estimate in the previous subsection is sharp:

**Theorem 4:** Assume that the functions $H_i, G_i$ are continuous, and such that the matrices $H(e^{j\omega})$ are unitary. Then

$$\| [K_1, \tau_1] \|_{\infty} = 1.$$  

(43)
Hence note that this observation does not preclude the symmetry condition (41). Since the columns of \( H(e^{j\omega}) \) are orthonormal, and\[\det(H(e^{j\omega})) = 1.\]
The minimum in (42) is therefore achieved by unitary filter banks, independent of the set of filters used. Relaxing the unitarity condition to biorthogonality will not decrease shift variance, but may well increase it, as stipulated by (42), and shown in Table I.

In a unitary FIR filter bank, the filters \( H_0(z), G_0(z) \) of the lowpass channel and \( H_1(z), G_1(z) \) of the highpass channel are commonly constructed from a FIR lowpass prototype \( H(z) \) of even length \( L \) by \( H_0(z) = H(z), \ G_0(z) = 2z^{-\frac{L-1}{2}}H(z^{-1}), \ H_1(z) = z^{-\frac{L-1}{2}}H(z^{-1}) \) and \( G_1(z) = -2H(z^{-1}) \) [15], [48]-[50]. Since the filter bank is unitary, we have \( \|K_1, \tau_1\|_\infty = 1. \) We now seek to derive additional information concerning near maximizers. Since the columns of \( \mathcal{H}(e^{j\omega}) \) are orthogonal, we obtain

\[
0 = e^{j(L-1)\omega} (H(e^{j\omega})H(-e^{-j\omega}) - H(-e^{j\omega})H(e^{-j\omega})) ,
\]
and therefore

\[
|H_0(-e^{j\omega})H_1(e^{j\omega})| = |H(-e^{j\omega})H(e^{-j\omega})| = |H_0(e^{j\omega})H_1(e^{-j\omega})| .
\]

Hence these filter banks fulfill the symmetry condition (41).

If \( h \) is in addition real-valued, we can establish that the maximum \( \lambda_1(e^{j\omega}) = 1 \) is attained for \( \omega = \pi/2. \) To this end, first observe that by assumption on \( h, \) it follows that \( |H(e^{j\omega})| = |H(-e^{j\omega})|; \) in particular for \( \omega = \pi/2. \) On the other hand, since \( \mathcal{H}(e^{j\pi/2}) \) is a unitary matrix, in particular

\[
1 = |H_0(e^{j\pi/2})|^2 + |H_1(e^{j\pi/2})|^2 = |H(e^{j\pi/2})|^2 + |H(-e^{j\pi/2})|^2 = |H(e^{j\pi/2})|^2 + |H(e^{-j\pi/2})|^2 = 2|H(e^{j\pi/2})|^2 .
\]

Hence \( |H(e^{j\pi/2})| = \sqrt{2}/2, \) and then (31) yields \( \lambda_1(e^{j\pi/2}) = 1. \)

Note that this observation does not preclude \( \lambda_1(e^{j\omega}) = 1 \) for \( \omega \) different from \( \pm \pi/2. \)

Hence for unitary filter banks arising from a prototype filter \( H \) with real-valued impulse response \( h, \) near maximizers of \( \|K_1, \tau_1\|_2 \) can be constructed by choosing \( s \) bandlimited to small intervals around \( \pm \pi/2. \)

### D. PR Filter Banks with Subband Processing

In this subsection, assume that \( K_0, K_1 \) are the channels of a PR filter bank, and consider the operator \( K = \alpha_0 K_0 + \alpha_1 K_1. \) The weights \( \alpha_i \) are complex factors applied to the channels, which can be viewed as the most elementary form of subband processing (cf. [23, p.250]).

**Theorem 5:** The operator \( K \) fulfills

\[
\|K, \tau_1\|_\infty = |\alpha_0 - \alpha_1||\|K_0, \tau_1\|_\infty .
\]

Shift variance as evaluated for a single channel therefore translates directly to the entire filter bank.

**Proof:** We analyse \( K \) by extending the notions developed for single channels, using \( [K, \tau_1] = \alpha_0 [K_0, \tau_1] + \alpha_1 [K_1, \tau_1]. \) For any input signal \( s \) with residual \( r = [K, \tau_1](s), \) we then obtain

\[
E_r[1] = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} s_M(e^{j\omega})B(e^{j\omega})s_M(e^{j\omega})d\omega
\]

where

\[
B(e^{j\omega}) = (\alpha_0 T_0(e^{j\omega}) + \alpha_1 T_1(e^{j\omega}))^* (\alpha_0 T_0(e^{j\omega}) + \alpha_1 T_1(e^{j\omega}))
\]

with \( T_1(e^{j\omega}) \) given by (30). In particular, the eigenvalues are given as

\[
|\sum_{i=0,1} \alpha_i G_i(-e^{j\omega}) H_i(e^{j\omega})|^2
\]

Substituting (36) into the first expression yields

\[
|\sum_{i=0,1} \alpha_i G_i(-e^{j\omega}) H_i(e^{j\omega})|^2
\]

which corresponds to straightforward scaling of the matrix norms for single channels in (21) by \( |\alpha_0 - \alpha_1|^2. \) This somewhat unexpected equality should be seen as an extension of (37) and (39). But then (44) is immediate.

Moreover, note that since \( T \) and \( T_1 \) have the same structure, any near maximizer \( s \) of \( \|K, \tau_1\|_2 \) is also a near maximizer of \( \|K_1, \tau_1\|_2, \) and vice versa. Of course, for \( \alpha_0 = \alpha_1, \) the aliasing components of the filter bank channels cancel each other out, making the output shift invariant.
IV. WIDE-BAND INPUT SIGNALS

The operator norm \(\| [K, \tau_m] \|_\infty \) characterizes the worst-case behavior of a filter bank or its channels, and also specifies the signals for which the worst case is almost reached in the form of near maximizers. Its basis is the matrix norm function \( \omega \mapsto \sqrt{\lambda_1(m, e^{j\omega})} \), whose peak height determines the operator norm. For unitary PR filter banks, near maximizers are narrow band signals around \( \pm \pi/2 \). In practice, one is often also interested in characterizing the shift variance of a filter bank for more representative signals, such as signals with a spectrum corresponding to an autoregressive process, a flat spectrum or, in images, spectra of signal models for lines or edges. In the following, we therefore derive worst case bounds under the assumption that the amplitude spectrum \( |S(e^{j\omega})| \) of an input signal \( s \) is given by some function \( \Phi(e^{j\omega}) \). Like the operator norm, these bounds are based on the matrix norm function \( \sqrt{\lambda_1(m, e^{j\omega})} \), but rather than being defined by the peak height, they correspond to a weighted integration of \( \sqrt{\lambda_1(m, e^{j\omega})} \). In addition to the peak height, they thus take also the shape of \( \sqrt{\lambda_1(m, e^{j\omega})} \) into account, which depends strongly on the filters used in a filter bank.

Specifying an amplitude spectrum via \( |S(e^{j\omega})| = \Phi(e^{j\omega}) \) also specifies the absolute values of the entries of the modulation vector \( s_M(e^{j\omega}) \) in (15). Therefore, with the weight function

\[
w(e^{j\omega}) = \sum_{m=0}^{M-1} |\Phi(W^m e^{j\omega})|^2
\]

formed from modulated versions of \( \Phi(e^{j\omega}) \), we define an additional measure of shift variance by

\[
\tilde{E}_w[m] = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \lambda_1(m, e^{j\omega}) w(e^{j\omega}) d\omega \quad (46)
\]

This new quantity can be understood as a middle course between the quantity \( E[m] \), which explicitly depends on a fixed choice of the signal \( s \), and the uniform bound \( E[m] \) obtained by looking at the worst-case input signals. It assumes knowledge of the amplitude spectrum of \( s \), but no information about its phase spectrum, giving a worst-case estimate for all signals with the known amplitude spectrum.

In fact, we have the estimate

\[
\| [K, \tau_m] s \|_2^2 = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} s_M^* (e^{j\omega}) A(m, e^{j\omega}) s_M(e^{j\omega}) d\omega
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \lambda_1(m, e^{j\omega}) w(e^{j\omega}) d\omega = \tilde{E}_w[m]
\]

valid for all \( s \) satisfying \( |S(e^{j\omega})| = \Phi(e^{j\omega}) \).

Unlike for the operator norm \( E[m] \), however, it is not entirely clear when \( \tilde{E}_w[m] \) is a sharp bound. In principle, it can only be (approximately) sharp if the modulation vectors \( s_M(e^{j\omega}) \) are near maximizers in the sense that

\[
\| A(e^{j\omega}) s_M(e^{j\omega}) \|_2 \approx \| A(m, e^{j\omega}) \|_\infty \cdot \| s_M(e^{j\omega}) \|_2.
\]

At a first glance, this assumption does not seem to be realistic; in any case, it does not only depend on the modulus of \( s_M(e^{j\omega}) \), but also on its phase.

However, if \( K \) is the channel operator of a two-channel PR filter bank satisfying condition (41), such as a unitary filter bank, then \( A(e^{j\omega}) \) is \( \lambda_1(e^{j\omega}) \) times an isometry, and thus \( \| A(e^{j\omega}) v \|_2 = \lambda_1(e^{j\omega}) \| v \|_2 \) holds for every vector \( v \in C^2 \). As a consequence, we have

\[
\tilde{E}_w[1] = \| [K, \tau_1](s) \|_2^2
\]

for every signal \( s \) with \( |S(e^{j\omega})| = \Phi(e^{j\omega}) \), and \( \| [K, \tau_1](s) \|_2^2 \) as in (15). By choosing \( \Phi(e^{j\omega}) \) to be wide-band, criterion (46) then complements the uniform bound \( E[m] \), which for unitary PR filter banks led to narrow-band near maximizers.

V. RESULTS

An algorithm to compute the operator norm from (22) is straightforward to implement. The size of \( M \), in particular if \( M = 2 \), is usually moderate by comparison to signal lengths, and the computation of \( \lambda_1(m, e^{j\omega}) \) can therefore be carried out efficiently using standard methods. Moreover, for FIR filters the function \( \lambda_1(m, e^{j\omega}) \) is smooth and thus easy to maximize. The following figures show plots of the largest eigenvalue \( \lambda_1(e^{j\omega}) \) for unitary filter banks with real-valued filters, specifically: Johnston QM filters of lengths 8 and 16 [15], [51] and Smith & Barnwell’s PR conjugated quadrature filters (PR-CQF) of lengths 8 and 16 [15], [50], [52] (Fig. 3), the multiplierless PR QM filters of lengths 4, 6, and 8 [15] and the Haar wavelet (Fig. 4), and the Daubechies 10 and 30 wavelets, the Coiflet-5 filter and the Symmlet-8 filter (Fig. 5). Evidently, the maximum of \( \lambda_1(m, e^{j\omega}) \) is equal to one for all unitary filter banks and occurs for \( \omega = \pi/2 \), as stipulated by (43) and section III-C. In all, curve corresponding to unitary filters exhibit a single large peak at \( \pi/2 \). In particular, since the maximum is attained only at \( \pi/2 \), the quantities \( E[m] \) only possess near maximizers. The degree of concentration necessary to achieve a good near maximizer can be read off the width of the peaks, which, as expected, decreases when the filter length increases. Note that this width is also reflected in the quantity \( \tilde{E}_w[m] \). Moreover, as discussed in section III-B, the filters fulfill (41), therefore, both eigenvalues of the matrix \( T_1 \) coincide.

For biorthogonal filters, the function \( \lambda_1(m, e^{j\omega}) \) is shown in Fig. 6. These exhibit a somewhat different behavior: First, obeying (42) and the discussion in section III-B, their maximum is higher than that of unitary filter banks. Also, the eigenvalues do not coincide; we plotted the lower eigenvalue \( \lambda_2(e^{j\omega}) \) as dashed curves. Consequently, greater care has to be taken in choosing near maximizers. Moreover, the critical frequency range for near maximizers is not an interval centered around \( \pi/2 \), rather two intervals centered somewhat to the left and right of \( \pi/2 \). Again, the width of the curves decreases with increasing filter length.

Recall from subsection III-B that for both orthogonal and biorthogonal filter banks, the curves are identical for both channels, and from subsection III-D, that they also hold for
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Fig. 3. Larger eigenvalue over $\omega$ for the Johnston 16 (solid line), Johnston 8 (dashed line), Smith & Barnwell 16 (dash-dotted line) and 8 (dotted line) filters.

Table I contains the values of the uniform bound $E(1)$ and of the bound $\tilde{E}_w(1)$ in section IV. To capture the full bandwidth of each curve, we chose the weight function $w$ in (46) to be constant, corresponding to signals with an ideally flat amplitude spectrum. As the second column confirms, the uniform bound is equal to one for all unitary PR filter banks. For the Johnston filters, this bound deviates slightly from one. The reason is that these filters are linear phase, and thus provide only approximately perfect reconstruction. A similar observation holds for the (nonlinear phase) multiplierless filters of length 6. For the biorthogonal filter banks, the uniform bound $E(1)$ is always higher than the one for unitary filter banks. These results corroborate the observation that the uniform bound is predominantly dependent on the filter bank structure, such as orthogonality or biorthogonality, rather than on the specific filter set chosen. This is different for the amplitude-spectrum dependent bound $\tilde{E}_w(1)$. Generally, we observe that the bound is lower for longer filters in a filter family, as expected. For the Haar filters, performance is worst, while the Daubechies 30 filters — the longest in our test field — perform best. The Smith-Barnwell PR-CQ filters outperform the Johnston QM filters of same lengths. The multiplierless QM filter of length 8 is inferior to both the Smith-Barnwell filter and the Johnston filter of the same length. From the filters of length 10 or less, the Smith-Barnwell 8 filter is best, followed by the biorthogonal 9/7 and 6/10 filters. Being biorthogonal, the worst-case behavior of the latter is slightly inferior to that of any unitary filter bank.

VI. CONCLUSIONS

We have developed a framework for the quantitative analysis of shift variance in critically sampled multirate filter banks. It is based on the concept of assessing changes in the output signal when filter bank operator $K$ and shift operator $\tau_m$ are interchanged. We started out from the difference signal between the filter bank response to a shifted input and the shifted output. We expressed this difference (or residual) signal via the commutator $[K, \tau_m]$ introduced in section II. The shift-dependent energy $E_\tau[m]$ of the residual signal is then given by the $L^2$-norm $\| [K, \tau_m](s) \|_2^2$ (15) of the commutator. The basis of the subsequent derivations is the matrix norm $\| T \|_\infty$ (20), (21). It is obtained from the larger eigenvalue $\lambda_1(e^{j\omega})$ of $T^*T$ in a frequency domain relation between the modulation vector of the input signal and the commutator. The frequency-dependent matrix norm allowed the derivation of the operator norm $\| [K, \tau_m] \|_\infty$, which, for filter banks with FIR filters, is given by the maximum of the matrix norm (23). The operator norm $\| [K, \tau_m] \|_\infty$ constitutes a uniform upper bound for the energy of the residual signal over all input signals, and can thus be regarded as quantifying the worst case behavior of a filter bank channel. While these considerations hold for general filter
the transition frequencies $\pm \pi/2$. Biorthogonal filter banks exhibit a somewhat different behavior, with the commutator norm being greater than one, and the near maximizers being narrow-band signals away from $\pm \pi/2$. We also considered PR filter banks with elementary subband processing in our framework, and showed that in this case, the upper bound (44) is proportional to the one for its channels.

In practice, one is often also interested in comparing shift variance of different filter banks for signals with given representative spectral properties. We therefore extended our framework by developing an additional measure (46) which bounds the commutator norm over signals with given amplitude spectra. Its basis is the same matrix norm $\|T\|_\infty$ as already used for the uniform bound. For unitary filter banks, it was shown that this bound is sharp. By selecting the input amplitude spectrum to be flat, i.e., wide-band, this measure complements the uniform bound with its narrow-band near maximizers. Detailed results for a variety of filters were provided and discussed. Naturally, this analysis is not restricted to signals with flat amplitude spectra, but could in the same way be carried out for signals representing structures critical for, say, the subjectively perceived quality of video data, such as lines or edges. This would permit a certain assessment of the effects of shift variance on visual image quality, albeit still based on the squared error metric rather than on perceptual quality measures, such as [53]–[55] for static images.

While we have restricted ourselves here to two-channel filter banks, the basic approach developed in section II pertains also to uniform $M$-band filter banks. The matrix $T(m, z)$ in (11) then is a $M \times M$ matrix of rank 2. It remains, though, still to be investigated whether its non-zero eigenvalues permit to derive a bound as in (42), and whether the commutator norms will be identical for all channels, as in (38) and (39) for the two-channel case. Further aspects are to extend the results towards $M$-band filter banks composed hierarchically from two-channel banks, and to LPSV systems in general which, as already stated in the introduction, can be regarded as critically sampled polyphase filter banks [3].

### References


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